Signature of Invigilators

1. ....

2. ....

MATHEMATICAL SCIENCES
Paper III

Roll No. ☐☐☐☐☐

(In figures as in Admit Card)

Roll No. ....

....

(In words)

JY—04/1

Name of the Areas/Section (if any)......................

Time Allowed : $2\frac{1}{2}$ Hours]

[Maximum Marks : 200

Instructions for the Candidates

1. Write your Roll Number in the space provided on the top of this page.
2. Write name of your Elective/Section if any.
3. Answer to short answer/essay type questions are to be written in the space provided below each question or after the questions in test booklet itself. No additional sheets are to be used.
4. Read instructions given inside carefully.
5. Last page is attached at the end of the test booklet for rough work.
6. If you write your name or put any special mark on any part of the test booklet which may disclose in any way your identity, you will render yourself liable to disqualification.
7. Use of calculator or any other Electronics Devices are prohibited.
8. There is no negative marking.
9. You should return the test booklet to the invigilator at the end of the examination and should not carry any paper outside the examination hall.

પરીક્ષાર્થીઓ માટે સૂચનાઓ :

૧. આ પૃષ્ઠના ઉપલા ભાગે આપેલી જગ્યામાં તમારી ક્રમાંક સંખ્યા (રોલ નંબર) લખો.
૨. તમે જે વિકલ્પનો ઉત્તર આપો તેનો સ્પષ્ટ નિર્દેશ કરો.
૩. ટૂંક નોંધ કે નિબંધ પ્રકારના પ્રશ્નોના ઉત્તર દરેક પ્રશ્નની નીચે આપેલી જગ્યામાં જ લખો. વધારાના કોઈ કાગળનો ઉપયોગ કરશો નહીં.
૪ અંદર આપેલી સૂચનાઓ ધ્યાનથી વાંચો.
૫. આ ઉત્તરપોથીને અંતે આપેલું પૃષ્ઠ કાચા કામ માટે છે.
૬. આ ઉત્તરપોથીમાં કયાંય પણ તમારી ઓળખ કરાવી દે એવી રીતે તમારું નામ કે કોઈ ચોકકસ નિશાની કરી હશે તો તમે આ પરીક્ષા માટે ગેરલાયક સાબીત થશો.
૭. કેલકયુલેટર અથવા ઈલેકટ્રોનિકસ સાધનો જેવાનો ઉપયોગ કરવો નહીં.
૮. નકારાત્મક ગુણાંક પદ્ધતિ નથી.
૯. પ્રશ્નપત્ર લખાઈ રહે એટલે આ ઉત્તરપોથી તમારા નિરીક્ષકને આપી દેવી. પરીક્ષાખંડની બહાર કોઈ પણ પ્રશ્નપત્ર લઈ જવું નહીં.

FOR OFFICE USE ONLY
Marks Obtained

Question Number	Marks Obtained	Question Number	Marks Obtained	Question Number	Marks Obtained
1.		20.		39.	
2.		21.		40.	
3.		22.			
4.		23.			
5.		24.			
6.		25.			
7.		26.			
8.		27.			
9.		28.			
10.		29.			
11.		30.			
12.		31.			
13.		32.			
14.		33.			
15.		34.			
16.		35.			
17.		36			
18.		37.			
19.		38.			

Total Marks Obtained..........................

Signature of the co-ordinator..............
(Evaluation)

MATHEMATICAL SCIENCES

PAPER-III

Note :—(*i*) This paper contains **forty (40)** questions, each carrying **twenty (20)** marks. The first **twenty (20)** pertain to mathematics, the remaining to statistics.

(*ii*) Attempt any **ten** questions.

(*iii*) Answer each question in not more than **300** words.

1. (*a*) Let $f(x) = x^2$ for $-\pi < x \leq \pi$ and $f(x + 2\pi) = f(x)$ for all $x \in \mathbf{R}$. Find the Fourier series of f. Hence or otherwise prove that :

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}.$$

(*b*) Let $f : \mathbf{R} \to \mathbf{R}$. Suppose that f is integrable on any bounded closed interval of $\mathbf{R}$. Then check the validity of the following statement and justify your answer.

$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{r\to\infty} \int_{-r}^{r} f(x)\,dx$, whenever the limit on the R.H.S. exists.

2. Find a complex number z such that :

$$|\sin z| > 10.$$

3. Let F be the field $\{0, 1, 2, ..., 6\}$ addition and multiplication being (mod 7). Prove that $x^2 + 4$ is irreducible in F[x]. Find the number of elements in the finite field $F[x]/(x^2 + 4)$.

4. (*a*) State Fatou's lemma.

(*b*) State and prove the monotone convergence theorem.

(*c*) Let f be a non-negative function which is measurable on a measurable subset E of $\mathbf{R}$. For each positive integer n, define

$$f_n(x) = \min(f(x), n) \quad (x \in E).$$

8. (*a*) Show that every separable metric space has a countable basis.

(*b*) Let $D = \{z \in \mathbb{C} \mid (z) < 2\}$. Suppose that f is analytic in D and has no zero in D. If $|f(z)| = 1$ for all z such that $|z| = 1$, prove that $f(z) = 1$ for all z in D.

9. (*a*) Prove that the following Boolean expressions are equivalent to one another :

(*i*) $(x \oplus y) * (x' \oplus z) * (y \oplus z)$

(*ii*) $(x * z) \oplus (x' * y) \oplus (y * z)$

(*iii*) $(x \oplus y) * (x' \oplus z)$

(*b*) Draw *two* graphs with *six* vertices one of which is Hamiltonian but not Eulerian and other is Eulerian but not Hamiltonian. Justify your claim.

10. (*a*) Find the Gaussian and mean curvatures for the surface

$$\vec{r} = (u\cos\theta, u\sin\theta, b\theta)$$

(b is a non-zero constant).

(*b*) What is a minimal surface ? What is the motivation for calling such a surface minimal ? Give *two* examples of minimal surfaces.

11. Let

$$J_\alpha(x) = x^\alpha \sum_{r=0}^{\infty} (-1)^r \frac{x^{2r}}{2^{2r+\alpha}\, r!\, \overline{|r + \alpha + 1}},$$

be the Bessel function of the first kind of order α, where $\alpha \geq 0$. Prove that :

(*a*) $\frac{d}{dx}(x^\alpha J_\alpha(x)) = x^\alpha J_{\alpha-1}(x)$

(*b*) $J_{\alpha-1}(x) + J_{\alpha+1}(x) = \frac{2\alpha}{x} J_\alpha(x),\ x \neq 0.$

Prove that :

$$\lim_{n \to \infty} \int_E f_n = \int_E f.$$

5. Define the characteristic of a field and if a field has non-zero characteristic, prove that its characteristic must be a prime number. Give an example of an infinite field with a finite (non-zero) characteristic.

6. (*a*) Show that if T is a one-to-one continuous linear transformation on a Banach space onto itself, then T^{-1} is continuous.

(*b*) Define a positive operator on a Hilbert space H. Prove that if P is a positive operator on H, then I + P is invertible. Deduce that I + T*T is invertible for every bounded operator T on H.

(*c*) State which of the following statements are true. Justify your answers :

(*i*) The closed unit ball of a normed linear space is convex.

(*ii*) The closed unit ball of a Banach space is compact.

(*iii*) If X is a normed linear space and Y is a proper subspace of X, then the interior of Y is empty.

(*iv*) If P is a projection on a Hilbert space H with range M and null space N and if P is self-adjoint, then $M \perp N$.

(*v*) In every Banach space, the parallelogram law holds.

7. Let X be a connected space and f and g be real valued continuous functions on X. Let $Z_f = \{x \in X \mid f(x) = 0\}$, with Z_g defined similarly.

Suppose that $Z_f \cap Z_g = \phi$. Prove that if $fg = 0$ on X, then $f = 0$ on X or $g = 0$ on X.

Give examples to show that the result does not hold when :

(*i*) X is connected, f, g are continuous, but $Z_f \cap Z_g \neq \phi$.

(*ii*) X is connected, $Z_f \cap Z_g = \phi$ but f or g is not continuous.

(*iii*) X is not connected, $Z_f \cap Z_g = \phi$ and f, g are continuous.

16. Derive generalized Bernoulli's equation from Euler's equation of motion of a perfect fluid assuming that the body forces are conservative, pressure is a function of density alone and flow is irrotational.

17. Find the stationary function of

$$\int_0^4 (xy' - y'^2)\,dx,$$

which is determined by the boundary condition $y(0) = 0$, $y(4) = 3$.

18. Using the method of successive approximations, solve the integral equation :

$$y(x) = x - \int_0^x (x - t)\,y(t)\,dt.$$

19. Find the order of convergence for the Newton-Raphson method. Show that the initial approximation x_0 for finding 1/N, where N is a positive integer, by the Newton-Raphson method must satisfy $0 < x_0 < \frac{2}{N}$, for convergence.

20. Solve

$$\frac{\partial^4 z}{\partial x^4} + \frac{\partial^2 z}{\partial y^2} = 0, (-\infty < x < \infty, y > 0),$$

satisfying the conditions :

(i) z and its partial derivatives tend to zero as $x \to \pm\infty$;

(ii) $z = f(x), \frac{\partial z}{\partial y} = 0$ on $y = 0$.

(Hint : Use Fourier transform. You may take

$$Z(\xi, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} z(x, y)\,e^{i\xi x}\,dx.$$

21. Explain weaknesses of the simplex method and illustrate how they are taken care of in Revised Simplex Method (RSM). Give computational procedure of RSM.

22. (a) Define a measurable function.

(b) Show that measurability of a function is with respect to a specified σ field.

12. Solve the Cauchy problem

$$(y + 2ux)\frac{\partial u}{\partial x} - (x + 2uy)\frac{\partial u}{\partial y} = \frac{1}{2}(x^2 - y^2),$$

$u(x, y) = 0$ on the straight line $x - y = 0$.

13. Find the smallest prime p which can be expressed in each of the forms $x^2 + y^2$, $x^2 + 2y^2$ and $x^2 + 3y^2$.

14. (*a*) Introduce holonomic and non-holonomic systems and write down Lagrange's equations of motion for a holonomic, conservative dynamical system with n degrees of freedom (no derivation).

(*b*) A particle of mass m moves on a smooth horizontal circular wire of radius a, which is free to rotate about a vertical axis through a point O, distant c from the centre C. If θ is the angle between CO and the radius to the particle, show that the kinetic and potential energies T, V, respectively, are given by

$$T = \frac{1}{2}M(a^2 + c^2)\omega^2 + \frac{1}{2}m\{c^2\omega^2 + a^2(\omega + \dot{\theta})^2 - 2ac\omega(\omega + \dot{\theta})\cos\theta\}$$

V = const., where M is the mass of the wire.

Hence show that the Lagrangian equation for the coordinate θ is :

$$a\ddot{\theta} + \dot{\omega}(a - c\cos\theta) = c\omega^2 \sin\theta,$$

where ω is the angular velocity of the wire.

15. State the necessary and sufficient conditions for the existence of a single valued displacement in three-dimensions. What is the total number of compatibility conditions for strain components ? Out of these equations, how many are algebraically independent ? How many compatibility conditions exist in two-dimensions ? Write them. If the strain components e_{ij} $(i, j = 1, 2, 3)$ are given by

$$e_{11} = x_1x_2,\ e_{22} = x_1^2,\ e_{12} = x_1x_2,\ e_{33} = e_{23} = e_{31} = 0,$$

then determine whether such a distribution is possible.

27. A draws 3 balls from Box 1 which contains 3 white and 2 red balls. B draws 3 balls from Box 2 which contains 2 white and 3 red balls. Let X and Y denote the number of white balls and red balls drawn by A and B respectively without replacement. Then :

(*a*) show that X and Y are i.i.d. r.v.s.

(*b*) find the joint distribution of U and V, where

$$U = \min (X, Y) \text{ and } V = \max (X, Y).$$

28. Define MLR property. Hence or otherwise derive a UMP test of size α for testing $H_0 : \theta \leq \theta_0$ against $H_1 : \theta > \theta_0$ based on a sample of size n from one parameter exponential family with parameter θ.

29. Define Sequential Probability Ratio Test (SPRT). Show that SPRT terminates with probability one. State the assumptions required.

30. Let $(x_1, y_1), (x_2, y_2), \ldots\ldots\ldots (x_n, y_n)$ be a random sample from the bivariate normal distribution $N(\mu, \mu, \sigma, \sigma, \rho)$:

(*i*) Give *two* unbiased estimators of μ.

(*ii*) Find a sufficient statistic for μ.

Is the sufficient statistic complete ? Justify.

(*iii*) Consider the class of estimators

$$\hat{\mu}(\alpha) = \alpha\overline{x} + (1 - \alpha)\overline{y}.$$

Find α so that $v(\hat{\mu})$ is minimum.

$$\left(\text{Here } \overline{x} = \frac{1}{n}\sum_{1}^{n} x_i \text{ and } \overline{y} = \frac{1}{n}\sum_{1}^{n} y_i\right).$$

31. Define Karl Pearson's test of goodness of fit. Obtain its large sample distribution.

(c) Also show that a continuous function of a real valued measurable function is a measurable function.

23. (a) State Liapunov's condition for a sequence of independent r.v.s. to obey the CLT.

(b) Let $\{X_n\}$ be a sequence of independent r.v.s. with

$$P(X_n = -n^{\theta}) = \frac{1}{2n^{\lambda}} = P(X_n = n^{\theta})$$

$$P(x_n = 0) = 1 - \frac{1}{n^{\lambda}}$$

where $\lambda > 0$.

Find conditions on θ and λ such that $\{X_n\}$ obeys the CLT.

24. Suppose $X_1, X_2, \ldots\ldots X_n$ are i.i.d. r.v.s. with $EX_1 = \mu$ and $V(X_1) = \sigma^2 < \infty$. Find the limit distribution of

$$\frac{\sqrt{n}\,(X_1 + \ldots\ldots + X_n - n\mu)}{\sum_1^n (X_i - \mu)^2}$$

(You have to state precisely any results that you wish to use).

25. (a) Stating the assumptions clearly, give (without proof) a formula to find EX^r in terms of the c.f. of X.

(b) Examine whether $\phi(t) = \exp\{-|t|^3\}$ is a c.f.

(c) Show that a r.v. is degenerate if and only if its variance is zero.

26. (a) State decomposition theorem for a d.f.

(b) Decompose the following d.f. F :

$$F(x) = \begin{cases} 0 & \text{if } x < -2 \\ \dfrac{x+3}{4} & \text{if } -2 \le x < -1 \\ 3/4 & \text{if } -1 \le x < 0 \\ 1 - \dfrac{1}{4}\exp(-x) & \text{if } x \ge 0 \end{cases}$$

38. (*a*) Define a homogeneous Poisson process $\{X(t), t \geq 0\}$ and obtain $P(X(t) = n)$ for $n \geq 0$.

(*b*) If the customers arrive at a service station in a Poisson fashion at the rate of 4 per hour and if the service station opens at 6 A.M., find the following :

$$P(X(t_2) = 5 | X(t_1) = 2)$$

$$P(X(t_1) = 2 | X(t_2) = 5)$$

where $6 < t_1 < t_2$.

39. Describe the construction of $(\overline{X}, R)$ control charts to control the future production. Obtain the expressions for their OC functions.

40. For a multi-item ($n \geq 2$ items) EOQ model for the warehousing problem (storage limitations), under the assumption of no shortages :

(*i*) Write mathematical model representing the inventory situation; and

(*ii*) write down steps for optimal solution of the model.

32. Drive the null distribution of sample correlation coefficient. Hence or otherwise suggest a test statistic for testing

$$H_0 : \rho = 0 \text{ against } H_1 : \rho \neq 0.$$

33. Let $\underline{X}_{3\times1} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$ follow $N_3(\underline{\mu}, \Sigma)$

Obtain the

(*i*) marginal distribution of (X_1, X_2);

(*ii*) conditional distribution of $(X_1/X_2, X_3)$.

34. Give a standard Gauss-Markov model and in its relation define (*i*) estimability of a linear parametric function (l.p.f.) (*ii*) best linear unbiased estimator (blue) of a l.p.f. Obtain BLUE and variance-covariance matrix of a l.p.f. Further show that BLUE is always unique.

35. What do you understand by double sampling or two phase sampling ? Obtain the double sampling ratio estimator for population mean. Obtain its bias and variance, when the second sample is a subsample from the first sample.

36. Explain the concepts of confounding and fractional replication.

Explain and illustrate for each of these :

(*i*) suitable situation for its use

(*ii*) the cost to be paid for it.

37. (*a*) Define a stationary distribution of a M.C.

(*b*) Does a stationary distribution always exist ? When it does, is it unique ?

If your answers are 'Yes', you have to prove. If the answers are 'No', give suitable examples.

Q. No. ....