Signature of Invigilators

1. ....

2. ....

Roll No. | | | | | |

(In figures as in Admit Card)

MATHEMATICAL SCIENCES

Paper III

Roll No. ....

....

(In words)

D/03/1

Name of Areas/Section (if any) ....

Time Allowed : $2\frac{1}{2}$ Hours]

[Maximum Marks : 200

Instructions for the Candidates

1. Write your Roll number in the space provided on the top of this page.
2. Write name of your Elective/Section if any.
3. Answer to short answer/essay type questions are to be written in the space provided below each question or after the questions in test booklet itself. No additional sheets are to be used.
4. Read instructions given inside carefully.
5. Last page is attached at the end of the test booklet for rough work.
6. If you write your name or put any special mark on any part of the test booklet which may disclose in any way your identity, you will render yourself liable to disqualification.
7. Use of calculator or any other Electronics Devices are prohibited.
8. There is no negative marking.
9. You should return the test booklet to the invigilator at the end of the examination and should not carry any paper outside the examination hall.

પરીક્ષાર્થીઓ માટે સૂચનાઓ :

૧. આ પૃષ્ઠના ઉપલા ભાગે આપેલી જગ્યામાં તમારી ક્રમાંક સંખ્યા (રોલ નંબર) લખો.

૨. તમે જે વિકલ્પનો ઉત્તર આપો તેનો સ્પષ્ટ નિર્દેશ કરો.

૩. ટૂંક નોંધ કે નિબંધ પ્રકારના પ્રશ્નોના ઉત્તર દરેક પ્રશ્નની નીચે આપેલી જગ્યામાં જ લખો. વધારાના કોઈ કાગળનો ઉપયોગ કરશો નહી.

૪. અંદર આપેલી સૂચનાઓ ધ્યાનથી વાંચો.

૫. આ ઉત્તરપોથીને અંતે આપેલું પૃષ્ઠ કાચા કામ માટે છે.

૬. આ ઉત્તરપોથીમાં કયાંય પણ તમારી ઓળખ કરાવી દે એવી રીતે તમારું નામ કે કોઈ ચોકકસ નિશાની કરી હશે તો તમે આ પરીક્ષા માટે ગેરલાયક સાબીત થશો.

૭. કેલકયુલેટર અથવા ઈલેક્ટ્રોનિકસ સાધનો જેવા ઉપયોગ કરવો નહીં.

૮. નકારાત્મક ગુણાંક પદ્ધતિ નથી.

૯. પ્રશ્નપત્ર લખાઈ રહે એટલે આ ઉત્તરપોથી તમારા નિરીક્ષકને આપી દેવી. પરીક્ષાખંડની બહાર કોઈપણ પ્રશ્નપત્ર લઈ જવું નહીં.

FOR OFFICE USE ONLY
Marks Obtained

Question Number	Marks Obtained	Question Number	Marks Obtained	Question Number	Marks Obtained
1		26			
2		27			
3		28			
4		29			
5		30			
6		31			
7		32			
8		33			
9		34			
10		35			
11		36			
12		37			
13		38			
14		39			
15		40			
16					
17					
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20					
21					
22					
23					
24					
25					

Total Marks Obtained................................

Signature of the co-ordinator....................

(Evaluation)

SEAL

MATHEMATICAL SCIENCES

PAPER-III

Note :—(*i*) This paper contains **forty (40)** questions, each carrying **twenty (20)** marks. The first **twenty (20)** questions pertain to mathematics, the remaining to statistics.

(*ii*) Attempt any **ten** questions.

(*iii*) Answer each question in about **300** words (**3** pages).

1. (*a*) Let ϕ be a continuous function on [0, 1]. Then prove that $\{f_n\}$, where

$$f_n(x) = x^n \phi(x) \text{ for all } x \in [0, 1],$$

converges uniformly on [0, 1] if and only if $\phi(1) = 0$.

(*b*) Let $E \subseteq \mathbf{R}^n$ be an open set. When do we say that a vector valued function $F : E \to \mathbf{R}^m$ is differentiable ?

Let $E = \mathbf{R}^2 - \{(0, 0)\}$ and define $F : E \to \mathbf{R}^2$ as $F = (f_1, f_2)$, where

$$f_1(x, y) = \frac{x^2 - y^2}{x^2 + y^2}, \quad f_2(x, y) = \frac{x^3}{x^2 + y^2}$$

for all $(x, y) \in E$. Check if F is in $C^1(E)$. State the result you use.

2. Obtain the first three terms of the Taylor's series expansion in powers of $(z - 1)$ of the function $\frac{1}{1 + z^2}$.

3. Let G be a region in the complex plane. Suppose a function f is analytic on G and there is $a \in G$ such that $|f(a)| \leq |f(z)|$ for all $z \in G$. Prove that either $f(a) = 0$ or f is constant.

4. Let S_n be the symmetric group on n symbols and A_n the corresponding alternating group. If a and b belong to S_n but not to A_n, prove that $ab \in A_n$.

5. (*a*) Let f be a real-valued function on **R**. Prove that $\{x \in \mathbf{R} \mid f(x) > \alpha\}$ is measurable for every α in **R** if and only if $\{x \in \mathbf{R} \mid f(x) > r\}$ is measurable for every rational r.

(*b*) Let E be a measurable subset of **R** of positive Lebesgue measure and let f be a non-negative measurable function on E. If $\int_E f(x)\,dx = 0$, prove that $f(x) = 0$ for almost all x in E.

6. Construct a field of 9 elements.

7. (*a*) Let X and Y be Banach spaces and T be a bounded operator from X to Y. Let N be the null space and R the range of T. Show that :

   (*i*) N is a closed subspace of X.

   (*ii*) $\Phi : \frac{X}{N} \to R$ defined by

   $\Phi(x + N) = T(x)$ is a bounded linear operator from $\frac{X}{N}$ onto R, which is one-to-one.

   (*iii*) Φ is a homeomorphism if and only if R is closed in Y.

   (*b*) Let H be a complex Hilbert space and $\{e_1, e_2, e_3, \ldots\ldots\}$ be a countable orthonormal set in H. Let $\{\alpha_n\}$ be a sequence of complex numbers such that

$$\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty.$$

   Show that

$$\sum_{n=1}^{\infty} \alpha_n e_n$$

   converges in H and if

$$x = \sum_{n=1}^{\infty} \alpha_n e_n,$$

   then $\alpha_n = (x, e_n)$, for each n.

8. (*a*) Define a locally connected space and give an example of a locally connected space which is not connected.

(b) Prove that every component of locally connected space is open.

(c) Prove that a compact locally connected space can have only a finite number of distinct components.

9. (a) Prove that every finite subset of a lattice has a glb and lub.

(b) Describe Königsberg bridge problem and draw a graph for the same. Using a suitable result from Graph theory show that it is not possible to walk over each of the seven bridges exactly once and return to the starting point. State the result you use.

10. Prove or disprove :

(a) If the Diophantine equation $a^2x + by = c$ has a solution, then the Diophantine equation $ax + by = c$ also has a solution.

(b) If the Diophantine equation $ax + by = c$ has a solution, then the Diophantine equation $a^2x + by = c$ also has a solution.

11. Prove that, at points of intersection of the surface $a(yz + zx + xy) = xyz$ with the sphere $x^2 + y^2 + z^2 = p^2$, the tangent plane to the surface makes intercepts on the coordinate axes, whose sum is constant.

12. (a) Define Lipschitz condition and hence discuss the existence and uniqueness of solution of the initial value problem :

$$\frac{dy}{dx} = y^{1/3}, \quad y(0) = 0.$$

(b) If $V(x, y, z)$ is a homogeneous function of degree n and satisfies Laplace's equation, then prove that :

$$\nabla^2(r^m V) = m(m + 2n + 1)r^{m-2}V.$$

13. (a) Solve the partial differential equation :

$$u_{xy} = -u_x.$$

(b) Find the eigenvalues and eigenfunction of the following Sturm-Liouville problem :

$$y'' + \lambda y = 0, \; y(0) = 0, \; y'(1) = 0.$$

14. For a certain dynamical system the kinetic and potential energies are :

$$T = \frac{1}{2}\{(1 + 2k)\dot{\theta}^2 + 2\dot{\theta}\dot{\phi} + \dot{\phi}^2\},$$

$$V = \frac{1}{2} n^2\{(1 + k)\theta^2 + \phi^2\},$$

where θ, ϕ are generalised coordinates and n, k are positive constants. Write down the Lagrange's equations of motion and deduce that :

$$(\ddot{\theta} - \ddot{\phi}) + n^2\left(\frac{1 + k}{k}\right)(\theta - \phi) = 0.$$

Hence prove that if $\theta = \phi, \dot{\theta} = \dot{\phi}$ at $t = 0$, then $\theta = \phi$ for all t.

15. (*i*) For an isotropic elastic material, find a relation between the first invariants of stress and strain; use this result to invert Hook's law so that strain is a function of stress.

(*ii*) With reference to an x-, y-, z-coordinate system, the matrix of a state of stress at a certain point of the body is given by

$$\underset{\sim}{T} = \begin{bmatrix} 2 & 4 & 3 \\ 4 & 0 & 0 \\ 3 & 0 & -1 \end{bmatrix}.$$

Find the stress vector and the magnitude of normal stress on a plane that passes through the point and is parallel to the plane

$$x + 2y + 2z = 6.$$

16. Test whether the motion specified by

$$\bar{q} = -\omega y\,\hat{i} + \omega x\,\hat{j} \quad (\omega = \text{const.})$$

is a possible motion for an incompressible fluid. If so, determine the equations of the streamlines.

Also test whether the motion is of the potential kind and if so determine the velocity potential.

17. Find the curve passing through (0, 0), which extremizes the integral

$$\int_0^{x_1} \frac{\sqrt{1 + y'^2}}{\sqrt{y}}\,dx$$

satisfying $y(x_1) = y_1$ (Brachistochrone problem).

18. Establish that the boundary value problem

$$\frac{d^2y}{dx^2} + \lambda y = 0$$

$$y(0) = 0, y(l) = 0,$$

where λ is a constant, is equivalent to the Fredholm equation :

$$y(x) = \lambda \int_0^l K(x, \xi)\ y(\xi)\, d\xi,$$

where

$$K(x, \xi) = \begin{cases} \dfrac{\xi(l - x)}{l}, & \xi < x \\ \dfrac{x(l - \xi)}{l}, & \xi > x \end{cases}$$

19. (a) For the following initial value problem, use Euler's modified method with $h = 0.1$ to obtain $y(1.2)$, and compare it with the exact solution :

$$y' = \frac{x - y}{x},\quad y(1) = 1.$$

(b) Set up a Newton iteration for computing the solution of equation :

$$2 \sin x = x.$$

Obtain the solution correct to four decimal digits.

20. (a) Find Laplace transform of

$$\frac{\sin x}{x},\quad x \neq 0.$$

(b) Solve the following initial value problem using Laplace transforms

$$y'' + 3y' + 2y = f(x);\ y(0) = 0 = y'(0)$$

where

$$f(x) = \begin{cases} 1, & 0 < x < a \\ 0, & x > a \end{cases}$$

21. Consider the L.P.P. :

Maximize : $Z = 5x_1 + 12x_2 + 4x_3$

Subject to :

$$x_1 + 2x_2 + x_3 \leq 10 \text{ and}$$
$$2x_1 - x_2 + 3x_3 = 8$$
$$x_1, x_2, x_3 \geq 0.$$

Find its dual and solve it (by any method). Also find the optimal solution for the given L.P.P.

22. (*a*) Suppose $\{A_n\}$ is a sequence of measurable sets in a measure space (Ω, A, μ). If $\Sigma \mu(A_n) < \infty$, show that :

$$\mu(\lim \sup A_n) = 0.$$

(*b*) If X is an exponential r.v., show that :

$$P(X > n \text{ i.o.}) = 0.$$

23. State and prove Kolmogorov's inequality.

24. (*a*) Define a Martingale.

(*b*) Let $\{Y_n\}$ be a sequence of i.i.d. r.v.s such that $Y_n \geq 0$ and $EY_n = 1$. Let

$$X_n = \prod_{k=1}^{n} Y_k \text{ and } Z_n = \sum_{k=1}^{n} Y_k - n.$$

Examine whether $\{X_n\}$ and $\{Z_n\}$ are martingales.

(*c*) Is

$$\left\{T_n = \sum_{1}^{n} Y_k\right\}$$

also a martingale ?

25. (*a*) State SLLN.

(*b*) State a sufficient condition for a sequence $\{X_n\}$ of independent r.v.s with $V(X_n) = \sigma_n^2 < \infty$ to obey SLLN.

(*c*) Examine whether SLLN holds for the sequence $\{X_n\}$ of independent r.v.s given by

$$P[X_n = \pm n^{\theta}] = \frac{1}{2n^2} \qquad \theta \in R$$

$$P[X_n = 0] = 1 - \frac{1}{n^2}.$$

26. Let the joint probability mass function of r.v.s. X and Y be defined by

$$f(x, y) = \begin{cases} \dfrac{C}{2^{x+y}} & y = x, x+1, x+2, \ldots\ldots \\ & x = 2, 3, 4, \ldots\ldots \\ 0 & \text{otherwise} \end{cases}$$

Find :

(*i*) the value of C

(*ii*) the marginal probability mass functions of X and Y

(*iii*) the conditional distribution of Y given $X = x$.

27. X is a random variable defined over a probability space $(\Omega, \mathbf{T}, P)$. A real valued function G is defined by

$$G(x) = P(X \leq x) \text{ for } x \in R.$$

Show that G is a cumulative distribution function.

Examine whether the function G defined as

$$G(x) = \begin{cases} 0 & \text{for } x < -1 \\ \dfrac{1+x}{9} & \text{for } -1 \leq x < 0 \\ \dfrac{2+x^2}{9} & \text{for } 0 \leq x < 2 \\ 1 & \text{for } x \geq 2 \end{cases}$$

is a cumulative distribution function. Obtain the values of the jumps of G at the points of discontinuity, if any. Also find $P(|X| \leq 1/2)$.

28. A simple random sample of size n is drawn from a normal distribution with mean zero and variance σ^2. In order to test the hypothesis $H_0 : \sigma^2 = 1$ against $H_1 : \sigma^2 > 1$; two test procedures are suggested.

Test 1 : Reject H_0 if $\frac{\Sigma x_i^2}{n} \geq C_1$

Test 2 : Reject H_0 if $\frac{\Sigma (x_i - \bar{x})^2}{(n-1)} \geq C_2$

C_1 and C_2 are chosen so that the two tests have the same level of significance α.

Determine C_1 and C_2 for given α and obtain the value of the power function of the two tests for $\sigma^2 = 2$.

29. The following data represent life times (hours) of batteries for two different brands :

Brand A :	40	30	40	45	55	30
Brand B :	50	50	45	55	60	40

Compute the Kolmogorov-Smirnov statistic to test H_0 that the population distribution of lengths of life for the two brands is the same.

30. Let T_1 and T_2 be two unbiased estimators having common variance $\alpha\sigma^2$ ($\alpha > 1$), where σ^2 is the variance of the UMVUE. Show that the correlation coefficient between T_1 and T_2 is at least $(2 - \alpha)/\alpha$.

31. Explain, what you understand by nth order statistic. Obtain the distribution of (*i*) median, (*ii*) range of a random sample of size n drawn from a population with density $f_X(x)$.

32. X_i's are i.i.d. r.v.s with $E(X_i) = 0$, $V(X_i) = \sigma^2$ and $E(X_i^4) = 3\sigma^4$. Obtain the asymptotic distribution of

$$Y_n = \sum_{i=1}^{n} X_i^2.$$

State precisely any result that you may use.

33. Let $\underline{X} \sim N_3(\underline{0}, \sigma^2 I_3)$. Consider the following quadratic forms :

$$Q_1 = (X_1^2 + X_2^2 + 2X_3^2 + 2X_1X_2)/2$$

$$Q_2 = (X_1^2 + X_2^2 - 2X_1X_2)/2$$

Examine whether Q_1/σ^2 and Q_2/σ^2 are distributed as Chi-square stating clearly the result you use. State the degrees of freedom in case either of them is Chi-square.

34. Explain how you would reduce the Gauss-Markov setup $(Y, X\beta, \sigma^2 G)$ to standard form, where G is a known positive definite symmetric matrix.

Establish necessary and sufficient condition for the estimability of linear parametric function in the standard Gauss-Markov setup.

35. Show that conventional ratio estimator is unbiased under Sen-Midzuno sampling scheme. Obtain the variance of this estimator.

36. Show that the necessary and sufficient condition for a block design to be connected is

$$\text{Rank } (c) = v - 1.$$

37. Consider the classical gambler's ruin problem where gambler A having an initial capital x plays against B whose initial capital is $(a - x)$. If A has a chance p of success in a single game and they play with one rupee as stake, find the probability of ruin of A. In this context, explain the statement "Expected gain of A is zero if the game is fair to both A and B".

38. (*a*) Define :

(*i*) long run distribution;

(*ii*) stationary distribution,

of a Markov Chain (M.C.).

(*b*) Examine whether a unique stationary distribution exists for an M.C. with TPM P given by

$$P = \begin{pmatrix} 1/2 & 0 & 1/2 & 0 \\ 1 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{pmatrix}.$$

Also obtain the long run distribution.

39. What is rectifying inspection ? How does it help in improving the quality of outgoing product ? Design a rectifying inspection single sampling plan with a specified AOQL.

40. (*a*) Explain what you understand by (r, s) inventory policy.

(*b*) The daily demand for a perishable item during a day occurs instantaneously over (0, 10). The holding cost of the item during the day is 50 paise and the unit penalty cost for running out of stock is Rs. 4.50. The unit purchase cost is 50 paise. A fixed cost of Rs. 5 is incurred each time an order is placed. Determine the optimal inventory policy for the item, if it exists.