Signature of Invigilators

1. ....

2. ....

Roll No. | | | | | |

(In figures as in Admit Card)

MATHEMATICAL SCIENCES

Paper III

Roll No. ....

....

(In words)

D—0102

Name of Areas/Section (if any) ....

Time Allowed : $2\frac{1}{2}$ Hours] [Maximum Marks : 200

Instructions for the Candidates

1. Write your Roll number in the space provided on the top of this page.
2. Write name of your Elective/Section if any.
3. Answer to short answer/essay type questions are to be written in the space provided below each question or after the questions in test booklet itself. No additional sheets are to be used.
4. Read instructions given inside carefully.
5. Last page is attached at the end of the test booklet for rough work.
6. If you write your name or put any special mark on any part of the test booklet which may disclose in any way your identity, you will render yourself liable to disqualification.
7. Use of calculator or any other Electronics Devices are prohibited.
8. There is no negative marking.
9. You should return the test booklet to the invigilator at the end of the examination and should not carry any paper outside the examination hall.

પરીક્ષાર્થીઓ માટે સૂચનાઓ :

૧. આ પૃષ્ઠના ઉપલા ભાગે આપેલી જગ્યામાં તમારી ક્રમાંક સંખ્યા (રોલ નંબર) લખો.

૨. તમે જે વિકલ્પનો ઉત્તર આપો તેનો સ્પષ્ટ નિર્દેશ કરો.

૩. ટૂંક નોંધ કે નિબંધ પ્રકારના પ્રશ્નોના ઉત્તર દરેક પ્રશ્નની નીચે આપેલી જગ્યામાં જ લખો. વધારાના કોઈ કાગળનો ઉપયોગ કરશો નહી.

૪. અંદર આપેલી સૂચનાઓ ધ્યાનથી વાંચો.

૫. આ ઉત્તરપોથીને અંતે આપેલું પૃષ્ઠ કાચા કામ માટે છે.

૬. આ ઉત્તરપોથીમાં ક્યાંય પણ તમારી ઓળખ કરાવી દે એવી રીતે તમારું નામ કે કોઈ ચોક્કસ નિશાની કરી હશે તો તમે આ પરીક્ષા માટે ગેરલાયક સાબીત થશો.

૭. કેલક્યુલેટર અથવા ઈલેક્ટ્રોનિક્સ સાધનો જેવા ઉપયોગ કરવો નહીં.

૮. નકારાત્મક ગુણાંક પદ્ધતિ નથી.

૯. પ્રશ્નપત્ર લખાઈ રહે એટલે આ ઉત્તરપોથી તમારા નિરીક્ષકને આપી દેવી. પરીક્ષાખંડની બહાર કોઈપણ પ્રશ્નપત્ર લઈ જવું નહીં.

FOR OFFICE USE ONLY
Marks Obtained

Question Number	Marks Obtained	Question Number	Marks Obtained	Question Number	Marks Obtained
1		26			
2		27			
3		28			
4		29			
5		30			
6		31			
7		32			
8		33			
9		34			
10		35			
11		36			
12		37			
13		38			
14		39			
15		40			
16					
17					
18					
19					
20					
21					
22					
23					
24					
25					

Total Marks Obtained..............................

Signature of the co-ordinator....................

(Evaluation)

SEAL

MATHEMATICAL SCIENCES
PAPER III

Note :— (*i*) This paper contains *forty* **(40)** questions, each carrying **twenty (20)** marks. The first *twenty* **(20)** questions pertain to mathematics, the remaining to statistics.

(*ii*) Attempt any *ten* questions.

(*iii*) Answer each question in about **300** words (3 pages).

1. (*a*) Let $f(x)$ be continuous on $[0, 1]$ such that $f(x) \geq 0$ for each x in $[0, 1]$. If $\int_0^1 f(x)\, dx = 0$, show that $f(x) = 0$ for each x in $[0, 1]$.

   (*b*) Show that $\int_\pi^\infty \frac{\sin x}{x}\, dx$ is convergent, but not absolutely convergent.

2. Find a complex number z such that $|z| < \frac{1}{50}$ and $\exp\left(\frac{1}{z}\right) = 1$.

3. If $f(z)$ is a non-constant function analytic in a domain D, continuous on $\overline{D}$ and has no zeros in $\overline{D}$, prove that $|f(z)|$ attains its minimum value on the boundary of D and not in its interior.

4. Let R be a commutative ring with unity. Find a necessary and sufficient condition that a homomorphic image of R should be a field.

5. (*a*) Show that every compact subset of a metric space is bounded.

   (*b*) Show that every compact subset of a Hausdorff space is closed.

   (*c*) Show that a closed and bounded subset of a metric space need not be compact.

   Give an example of a metric space in which every closed and bounded subset is compact.

6. Find the Galois group of the polynomial $x^2 + 1$ over the field of real numbers.

7. (*a*) Let X be a normed linear space. For x, y in X, define $\rho(x, y) = 0$ if $x = y$ and $\rho(x, y) = \| x - y \| + 1$ if $x \neq y$. Show that ρ is a metric on X. Also show that there is no norm $\| \|'$ on X such that $\| x - y \|' = \rho(x, y)$ for x, y in X.

(b) Let H be a separable Hilbert space. Show that any orthonormal set in H is atmost countable.

8. (a) Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of connected subsets of a topological space X such that $A_n \cap A_{n+1} \neq \phi$ for all n. Show that $\bigcup_{n=1}^{\infty} A_n$ is connected.

(b) Show that $E = \{(x, y) \mid x \in \mathbf{R}, y \in \mathbf{R}, x \text{ or } y \text{ is a positive integer}\}$ is a connected subset of $\mathbf{R}^2$.

9. Draw a connected graph of 7 vertices in which every nonzero distance is 1, 2, or 3.

10. (a) State the Serret-Frenet formulae, giving the meaning of each symbol, occurring in the formulae. Give a necessary and sufficient condition for a curve to be a plane curve, stating clearly the assumptions you are making. Prove your result.

(b) If the principal normal at each point on a curve C is a binormal to some other curve Γ at some point on Γ, prove that $k = c(\tau^2 + k^2)$, where k and τ are curvature and torsion at a point of C and c is a constant.

11. (a) Using the method of successive approximations, solve the integral equation

$$\varphi(x) = 1 + \int_0^x \varphi(t)\, dt,$$

taking $\varphi_0(x) \equiv 0$.

(b) Find the eigenvalues and eigen functions of the Sturm-Liouville problem :

$$y'' + \lambda y = 0,\ y(0) = 0,\ y(\pi) = 0.$$

12. Reduce the equation

$$u_{xx} - 4x\, u_{xy} + 4x^2 u_{yy} = 0$$

to canonical form.

13. For positive integers m, r, s we have $m = rs$ and m and r are sums of two squares. Prove that s is also a sum of two squares.

14. Show that the transformation given by the equation

$$P = \frac{1}{2}(p^2 + q^2),\ Q = \tan^{-1}\frac{q}{p}$$

is canonical. Find its generating function.

15. The strain tensor at a point is

$$E = \begin{pmatrix} 5 & -1 & -1 \\ -1 & 4 & 0 \\ -1 & 0 & 4 \end{pmatrix}.$$

Determine :

(i) The extensional strain at this point in the direction of $\hat{i} + \hat{k}$

(ii) Principal strains

(iii) Maximum normal strain

(iv) Maximum shearing strain and

(v) Strain invariants.

16. Test whether the motion specified by

$$\vec{q} = \left[\frac{kx}{(x^2 + y^2 + z^2)^{3/2}}, \frac{ky}{(x^2 + y^2 + z^2)^{3/2}}, \frac{kz}{(x^2 + y^2 + z^2)^{3/2}}\right](k = \text{constant}),$$

is a possible motion for an incompressible fluid. Also test whether the motion is of the potential kind.

17. (a) Find the extremals of the following functional

$$J(y(x)) = \int_0^1 (y'^2 + 4y^2)\,dx;\ y(0) = e^2,\ y(1) = 1.$$

(b) Translate the problem of determining an approximate solution of the Poisson equation

$$\Delta z = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = -1,$$

in the rectangle $-a \le x \le a$, $-b \le y \le b$, subject to the condition $z = 0$, on the boundary to a variational problem.

18. (a) Solve the following integral equation with degenerate kernel :

$$\varphi(x) - \lambda \int_0^{\frac{\pi}{2}} \sin x \cos t \, \varphi(t) \, dt = \sin x.$$

(b) Solve the equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = e^{-x} \cos y$$

which tends to zero as $x \to \infty$ and has the value $\cos y$ when $x = 0$.

19. (a) Design a Newton iteration for cube roots and compute $7^{1/3}$ correct to 2 decimal digits.

(b) The following data hold for a polynomial $p(x)$. Find its degree :

x =	−2	−1	0	1	2	3.
$p(x)$ =	−5	1	1	1	7	25.

20. (a) Solve using Laplace transform :

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 2 \cos x,$$

$$y(0) = 3, \left.\frac{dy}{dx}\right|_{x=0} = 0.$$

(b) Using Fourier integrals, show that :

$$\int_0^{\infty} \left(\frac{\cos \omega x + \omega \sin \omega x}{1 + \omega^2} \right) d\omega = \begin{cases} 0, & \text{if } x < 0 \\ \pi/2, & x = 0 \\ \pi e^{-x} & x > 0 . \end{cases}$$

21. Explain what is Gomory's cut method of solving an integer programming problem (IPP).

Give first step of iteration starting with the given solution for the LPP to solve the IPP under consideration :

Maximize $2x_1 + x_2 + 3x_3$

subject to

$$2x_1 + x_2 + x_3 \leq 17$$

$$x_1 + x_2 + 2x_3 \leq 15$$

x_1, x_2, x_3 are non-negative integers. Optimal solution for LPP is $\left(\frac{19}{3}, 0, \frac{13}{3}\right)$.

22. Solve the following :

Minimize

$$(x_1, x_2, x_3) \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

subject to

$$2x_1 + x_2 + x_3 \geq 4$$

$$x_1 + x_2 + 2x_3 \geq 6$$

$$x_1, x_2, x_3 \geq 0$$

23. (a) Define a measurable function.

(b) If f is a real-valued measurable function and g is a continuous function from R to R, show that $g \circ f$ is also a measurable function.

(c) Show that if f is a real-valued measurable function, $|f|$ is also a measurable function but the converse is not necessarily TRUE.

24. Define convergence in probability and in the rth mean.

Show that $X_n \to X$ in the rth mean if and only if

$$\frac{|X_n - X|^r}{1+|X_n - X|^r} \to 0 \text{ in probability.}$$

25. State Liapunov's central limit theorem (CLT). Find values of θ for which the CLT holds for the sequence of independent r.v.s $\{X_n\}$ given by

$$P\left[X_n = \pm n\right] = \frac{1}{2n^{\theta}}$$
$$P\left[X_n = 0\right] = 1 - \frac{1}{n^{\theta}}$$
$$n = 1, 2, \ldots, \theta > 0.$$

26. Let (X, Y) be jointly distributed with probability density function (p.d.f.)

$$f(x, y) = 2 \quad , 0 < x < y < 1$$
$$= 0 \quad \text{otherwise.}$$

Obtain the marginal p.d.f.s $f_X(x)$ and $f_Y(y)$ and conditional p.d.f.s $f_{Y|X}(y|x)$ and $f_{X|Y}(x|y)$. Hence find $V(X|y)$.

27. State Decomposition theorem for a distribution function.

Obtain the component d.f.s for the d.f.F given below :

$$F(x) = \begin{cases} 0 & \text{if } x < -1 \\ \dfrac{x+2}{4} & \text{if } -1 \le x < 0 \\ 3/4 & \text{if } 0 \le x < 1 \\ \left(1 - \dfrac{1}{4}e^{-(x-1)}\right) & \text{if } x \ge 1. \end{cases}$$

28. Obtain the characteristic function of the distribution with density function

$$f(x) = C.e^{-\lambda x} x^{p-1} \text{ for } x > 0, \lambda > 0, p > 0$$
$$= 0 \quad \text{otherwise.}$$

Hence deduce the characteristic function of a chi-square variable with n degrees of freedom.

29. Define UMVUE for a parametric function. Let $X_1, X_2, \ldots, X_n$ be a random sample from Poisson with parameter θ. Find UMVUE of $e^{-\theta}$. Explain the theorem used in it.

30. (a) Define :

(i) Risk function;

(ii) Prior distribution;

(iii) Posterior distribution;

(iv) Bayes risk.

(b) Let $X \sim$ Binomial (n, θ). Obtain Bayes estimator for θ, using squared error loss function, w.r.t. prior $g(\theta)$ given by

$$g(\theta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)}\,\theta^{\alpha-1}(1-\theta)^{\beta-1} \quad 0 < \theta < 1,$$
$$\alpha, \beta > 0.$$

31. Explain what you understand by UMP test. Let X have p.d.f. $f(x, \theta)$, which is a member of one parameter exponential family. Then show that a UMP test of size $\alpha (0 < \alpha < 1)$ always exists for testing $H_0 : \theta \leq \theta_0$ *Vs* $H_1 : \theta > \theta_0$.

32. On the basis of n independent observations from a distribution with density

$$f(x; \theta) = e^{-(x-\theta)} \quad , x \geq \theta$$
$$= 0 \qquad \text{otherwise,}$$

obtain the likelihood ratio test for testing $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$. If λ denotes the likelihood ratio, show that under H_0, $-2 \ln \lambda$ has an exact central chi-square distribution with two degrees of freedom.

33. If $\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_n$ is a random sample of size n $(n \geq p)$ from multivariate normal distribution $N_p(\underset{\sim}{o}, \Sigma)$, establish that

$$\frac{1}{n}\sum_{i=1}^{n} \underset{\sim}{x}_i \underset{\sim}{x}'_i$$

is maximum likelihood estimate of Σ.

34. Explain what is Gauss-Markov set up $(Y, X\beta, \sigma^2 I)$. When does one say that $p'\beta$ is estimable ?

For a RBD with 2 blocks and 2 treatments, write down the model, identify the design matrix X and establish that the difference between the two treatment effects is estimable.

35. Define Horvitz-Thompson's estimator for population total. Obtain its variance when sampling is carried out as per PPSWOR. How will you proceed to estimate this variance unbiasedly on the lines suggested by Yates and Grundy ? Discuss the limitation of this estimator, if any.

36. When is a block design called BIBD ? For a BIBD with b blocks, k plots in each block, v treatments, r replications and λ as the parameter showing the number of times any pair of treatments appears in the same block, establish

$$b \geq \max(v, v + r - k)$$

$$\lambda(v - 1) = r(k - 1).$$

37. Consider a M.C. $\{X_n\}$ with state space $\{1, 2, 3, 4\}$ and TPM

$$P = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{bmatrix} 1-\alpha & \alpha & 0 & 0 \\ \beta & 1-\beta & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix}$$

where $0 < \alpha, \beta < 1$.

Find the recurrence class C of the M.C.

Show that

$$P[X_n = 3 \text{ or } 4 \text{ for all } n \mid X_0 = 4] = 0.$$

What is $P[X_n \in C \text{ eventually} \mid X_0 = 4]$?

38. (a) Define a branching process $\{X_n\}$.

(b) If G(s) is the P.G.F. corresponding to a branching process with $X_0 = 1$, $0 < G(0) < 1$, obtain the probability of ultimate extinction of the process in terms of G(s).

(c) If $G(s) = \frac{1}{3} + \frac{1}{3}s^2 + \frac{1}{3}s^3$, find the probability of explosion of the process.

39. Explain the concept of hazard rate. Show that in the Weibull case hazard rate is increasing or decreasing depends upon the value of the shape parameter.

Determine whether the following distribution is IFR or DFR :

$$f(x) = \frac{1}{x^2}, \quad 1 < x < \infty.$$

40. What do you understand by quantity discounts ? Derive the EOQ formula for an inventory problem with ordering cost K, inventory carrying cost C_I per unit item for a month, demand rate D per month and the cost of one unit of item is c_1 if less than q_1 units are ordered, it is c_2 if at least q_1 but less than q_2 units are ordered and it is c_3 if at least q_2 units are ordered. Explain its implementation.

Q. No. ....